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1979 J. Phys. A: Math. Gen. 12 1309

(http://iopscience.iop.org/0305-4470/12/8/023)

J. Phys. A: Math. Gen., Vol. 12, No. 8, 1979. Printed in Great Britain

High-order behaviour of zero-component field theories without the $n \to 0$ limit

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Received 13 October 1978, in final form 18 December 1978

Abstract. We show that direct evaluation of the Green function of the random electron or polymer problem by a steepest-descents method yields results identical to those obtained in the $n \to 0$ limit of Landau–Ginzburg field theory. The essential feature of the method lies in the observation that only a restricted set of potnetials contributes to Im $G(x_1, x_2)$ and therefore the variational procedure must be restricted to this class.

Our calculation shows that the correct asymptotic information has been used in calculating critical exponents for the polymer system, and verifies the density of states

$$\rho(E) = \rho_0 |g|^{-(d+1)/2} (-E)^{d(5-d)/4} \exp[-c(-E)^{2-d/2}|g|^{-1}]$$

found recently by $n \to 0$ methods. In our method the expression for the degree of localisation is obtained immediately.

1. Introduction

It has been pointed out by several authors that the properties of disordered electronic systems (Ma 1972, Thouless 1975) and polymers in solution (de Gennes 1972, Emery 1975) may be discussed by taking advantage of the formal equivalence, order by order in perturbation theory, with the $n \to 0$ limit of an n-component Ginzburg–Landau field theory. Renormalisation group theory and the ϵ expansion can then be used to calculate exponents (de Gennes 1972) for the polymer problem. For the electron system the analogy has not proved as useful, because the only known fixed point is unattainable (Aharony *et al* 1976). For this problem a knowledge of perturbation theory beyond that given by the renormalisation group is certainly needed.

Recently it has been shown that, although perturbation theory in the coupling constant g has zero radius of convergence, the divergence is controlled by the existence of non-trivial solutions of the classical Ginzburg–Landau equations with finite-action instantons (Lipatov 1976, 1977, Langer 1967, Brézin *et al* 1977). The form of the divergence can be characterised and the imaginary part of the correlation functions can be computed exactly for small negative g. This information has been used to compute critical exponents to high accuracy (Le Guillou and Zinn-Justin 1977), and more

§ Supported in part by the Alexander von Humboldt–Stiftung and the NSF under Grant Nos DMR 76-80560 and DMR 73-02605.
‖ Supported by the Deutsche Forschungsgemeinschaft.

recently Cardy (1978) has calculated the density of localised states in the energy band of a disordered solid.

The question remains whether or not the analogy between field theory in the $n \to 0$ limit and disordered systems (polymer solutions) is retained when non-perturbative methods such as those discussed above are used. This can be judged only from the results. Cardy, using the $n \to 0$ limit, found the energy dependence for the density of states $\rho(E)$ in the tail of an infinite band:

$$\rho(E) = \text{constant} \times (-E)^{d(5-d)/4} \exp[-\text{constant} \times (-E)^{2-d/2}]. \tag{1.1}$$

The energy dependence of the exponent agrees with the earlier results of Halperin and Lax (1966, 1967), Zittartz and Langer (1966) and Edwards (1970). (Compare also Thouless and Elzain (1978), Edwards (1978).) The prefactor $E^{3/2}$ for $d = 3$ was first given by Halperin and Lax (1966, equation (7.16)). Edwards' method suggests a prefactor $E^{3/4}$, whereas Zittartz and Langer do not give the energy dependence of the prefactor explicitly for $d = 3$. All these approaches—except for Edwards (1978)—use variational methods without resorting to the analogy with $n \to 0$ field theory.

Thus it is not completely clear whether the $n \to 0$ method gives the correct energy dependence of $\rho(E)$; and whether it gives the correct proportionality constants in equation (1.1) is an open question. To solve these problems we here re-derive equation (1.1) using the recently established field theoretic methods, but avoiding the $n \to 0$ limit. Although the conventional steepest-descents method cannot be applied, we are able to formulate a variational principle which permits exact calculation of the density of states for $g \to 0$ and makes contact with the earlier work of Zittartz and Langer (1966). We recover the results of the $n \to 0$ limit, including the proportionality constants, and we thus justify the application of the $n \to 0$ method as given by Cardy.

2. The variational principle

We are concerned with the properties of the Green function

$$\mathcal{G}(x_1, x_2, [\psi]) = \langle x_1 | (-\tfrac{1}{2}\nabla^2 + \tfrac{1}{2}m^2 + \mathrm{i}\sqrt{g}\,\psi(x))^{-1} | x_2 \rangle \tag{2.1}$$

averaged over the (real) single-particle potential $\psi(x)$:

$$G(x_1, x_2) = \int \mathrm{D}[\psi] \exp\Big(-\int \mathrm{d}^d x \psi^2(x)\Big) \mathcal{G}(x_1, x_2, [\psi]). \tag{2.2}$$

Here d is the dimensionality of the space. For real positive values of g and $m^2 > 0$, $G(x_1, x_2)$ is known to represent correlations among the end-points of a self-interacting polymer molecule of length $N \sim m^{-2}$. For negative g and $\frac{1}{2}m^2 = -E \pm \mathrm{i}\delta$, $G(x_1, x_2)$ represents the average Green function for a particle of energy E moving in a random potential $-|g|^{1/2}\psi(x)$. For $m^2 \geqslant 0$, $G(x_1, x_2)$ has a cut in the complex g plane along the negative real axis; $G(x_1, x_2)$ is then fully determined by its imaginary part along this cut via a dispersion relation. Further, the limiting behaviour of the large orders of the formal perturbation series in powers of g are characterised by Im G for small negative values of g. We here aim at a calculation of Im G for $g < 0$, $|g| \to 0$.

The standard solution to this problem (Brézin *et al* 1977) exploits the observation that $G(x_1, x_2)$ can be represented formally as the $n = 0$ limit of a Landau–Ginzburg field

theory for an n-component spin field $S(x) = (S_1(x), \ldots, S_n(x))$:

$$G(x_1, x_2) = \lim_{n \to 0} \int \mathrm{D}[S] S_1(x_1) S_1(x_2) \times \exp\Big(- \int \mathrm{d}^d x [\tfrac{1}{2} m^2 S^2(x) + \tfrac{1}{2} (\nabla S(x))^2 + \tfrac{1}{4} g (S^2(x))^2] \Big). \tag{2.3}$$

A steepest-descents analysis of this functional integral yields the desired information as a function of n; then the limit $n \to 0$ is taken in the result. For the reasons explained above we wish to avoid the $n \to 0$ limit. Instead we apply the steepest-descents method directly to equation (2.2).

Such an approach has been used previously by several authors (Zittartz and Langer 1966, Edwards 1970) and has led both to a prediction of the density of localised states in the band tail of the random electron system and to a theory of the configuration of a polymer molecule in a good solvent. In this work $\psi(x)$ is varied to find that potential which yields the maximum contribution to the functional integral (2.2). However, we note that the form of the integrand is not directly suited to a steepest-descents evaluation, which may lead to incorrect results in the problem of interest to us. We illustrate this by a simple example.

Consider the one-dimensional integral

$$I_1 = \int_{-\infty}^{+\infty} \mathrm{d}\psi \, \mathrm{e}^{-\psi^2} (1 + \mathrm{i}\sqrt{g}\, \psi)^{-1}, \qquad g = -|g| + \mathrm{i}\delta, \tag{2.4}$$

which may be thought of as the propagator of a one-dimensional field defined at one point. The imaginary part of I_1 is found from

$$I_1 = -|g|^{-1/2} P \int_{-\infty}^{+\infty} \mathrm{d}\psi \, \mathrm{e}^{-\psi^2} (\psi - |g|^{-1/2})^{-1} + \mathrm{i}\pi |g|^{-1/2} \mathrm{e}^{-1/|g|}, \tag{2.5}$$

where P denotes the principal value. We now try to evaluate I_1 by the steepest-descents method. The saddle point ψ_s is found as the solution of the equation

$$2\psi + (\partial/\partial\psi) \ln(1 + \mathrm{i}\sqrt{g}\, \psi) = 0. \tag{2.6}$$

Therefore

$$\psi_s = -(1/\mathrm{i}\sqrt{g})(1 + \tfrac{1}{2} g + \mathrm{O}(g^2)). \tag{2.7}$$

Setting $\psi = \psi_s + \phi$ and expanding the integrand around the saddle point we find

$$I_1 = \mathrm{e}^{-\psi_s^2} (1 + \mathrm{i}\sqrt{g}\, \psi_s)^{-1} \int_{-\infty}^{+\infty} \mathrm{d}\phi \, \exp\Big[-\Big(1 + \frac{1}{g}\Big) \phi^2 + \mathrm{O}(\phi^3) \Big]. \tag{2.8}$$

For small negative g this integral has to be interpreted as the analytic continuation from $g > 0$, or it can be obtained by properly deforming the contour of integration (see e.g. Langer 1968, Wallace 1979). The result is

$$\mathrm{Im}\, I_1 \sim (\pi |g|)^{1/2} (1 + \mathrm{i}\sqrt{g}\, \psi_s)^{-1} \mathrm{e}^{-\psi_s^2}. \tag{2.9}$$

With ψ_s given by equation (2.7) we find

$$\mathrm{Im}\, I_1 \sim \exp(\pi^{1/2}) |g|^{-1/2} \mathrm{e}^{-1/|g|}, \tag{2.10}$$

which is to be compared with the correct result of equation (2.5),

$$\mathrm{Im}\, I_1 = \pi |g|^{-1/2} \mathrm{e}^{-1/|g|}. \tag{2.11}$$

The difference between these two results shows that the steepest-descents approximation incorrectly estimates Im I_1. Indeed, a closer inspection shows that all the higher-order terms in equation (2.8) contribute to the constant factor in front of the exponential $\exp(-1/|g|)$. We therefore expect that the straightforward steepest-descents evaluation of our full problem (equation (2.2)) could lead to incorrect factors multiplying the leading exponential.

This simple example, however, shows how to circumvent this problem. As we see from equation (2.4), the imaginary part of I_1 is given by the pole term; similarly in the full problem only those potentials for which $\mathcal{G}(x_1, x_2, [\psi])$ is singular contribute to the imaginary part. *Therefore we must restrict our variation to this class of functions.* This defines a surface in function space; we then ask for that function ψ_s on this surface which maximises the weight $e^{-\psi^2}$. We again illustrate the method by a simple example.

Consider the two-dimensional integral

$$G_{ij} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e^{-(x^2+y^2)} [A^{-1}]_{ij}, \tag{2.12}$$

where the matrix A is given by

$$A = \begin{bmatrix} i\sqrt{g}x & 1 \\ 1 & i\sqrt{g}\,y \end{bmatrix}. \tag{2.13}$$

The matrix elements G_{ij} are analytic in the cut g plane. We will again determine the imaginary part for $g = -|g| + i\delta$ exactly and by the variational method.

2.1. Exact calculation

The inverse matrix A^{-1} is given by

$$A^{-1} = \begin{bmatrix} i\sqrt{g}\,y & -1 \\ -1 & i\sqrt{g}\,x \end{bmatrix} \frac{-1}{gxy+1}. \tag{2.14}$$

Hence we see that the diagonal elements of G are integrals of odd functions of (x, y) and vanish identically. The off-diagonal elements are given by

$$G_{12} = G_{21} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{e^{-(x^2+y^2)}}{gxy+1}, \tag{2.15}$$

and therefore

$$\text{Im } G_{12} = \text{Im } G_{21} = \frac{2\pi}{|g|} \int_0^{\infty} \frac{dy}{y} e^{-y^2-(|g|y)^{-2}} = \frac{2\pi}{|g|} K_0\left(\frac{2}{|g|}\right), \tag{2.16}$$

where K_0 is a modified Bessel function. As we are only concerned with Im G_{12} for $|g|$ small, we may use the asymptotic expansion

$$K_0(z) \underset{z\to\infty}{\simeq} (\pi/2z)^{1/2} e^{-z}(1 + O(1/z)). \tag{2.17}$$

We find

$$\text{Im } G_{12} = \pi^{3/2}|g|^{-1/2} e^{-2/|g|}(1 + O(|g|)). \tag{2.18}$$

2.2. Variational calculation

Clearly G_{ij} has an imaginary part only if the matrix A has a vanishing eigenvalue. This defines a surface (line) in our two-dimensional space:

$$\det A = -(gxy + 1) = 0. \tag{2.19}$$

The maximum contribution to Im G_{ij} is due to that point (x_s, y_s) on the line defined by equation (2.19) which minimises the exponent $x^2 + y^2$. The variational condition is

$$\delta(x^2 + y^2) - \rho\delta(gxy + 1) = 0, \tag{2.20}$$

where ρ is a Lagrange multiplier for the condition (2.19). These equations are easily solved for the saddle-point solution:

$$x_s^2 = y_s^2 = |g|^{-1}. \tag{2.21}$$

Note there are two saddle points symmetrically placed about the origin. To evaluate the contribution from any one saddle point, we transform from the coordinates (x, y) to a set of generalised coordinates adapted to the geometry of our problem. As one coordinate we take that eigenvalue λ whose zero defines the surface. As the other coordinate any one of the pair (x, y) can be used. The Jacobian of the transformation $J = \mathrm{d}x/\mathrm{d}\lambda|_{y\ \text{fixed}}$ is found from the eigenvalue equation; we will only need the value of the Jacobian at the saddle point $\lambda = 0$, $y = y_s$. We find

$$J = -2|g|^{-1/2}. \tag{2.22}$$

We now expand the exponent $x^2 + y^2$ around the saddle point in the usual way, but we only allow variations in the surface; thus

$$1 + g(x_s + \delta x)(y_s + \delta y) = 0; \tag{2.23}$$

that is,

$$\delta x = -y_s\delta y/(y_s + \delta y). \tag{2.24}$$

Finally we find

$$\text{Im } G_{12} = -\pi\int \mathrm{d}\lambda\ \delta(\lambda) \int \mathrm{d}(\delta y)|\,\mathrm{d}x/\mathrm{d}\lambda|\,\mathrm{e}^{-2y_s^2-4(\delta y)^2}|\phi_0\rangle\langle\phi_0|. \tag{2.25}$$

Here ϕ_0 is the eigenstate of zero eigenvalue of the matrix A evaluated at the saddle point. The integration over λ gives the contribution to Im G from the zero eigenvalue, and the integration over δy collects the contributions from the surface $\lambda = 0$. For the diagonal elements the contribution of the two saddle points at $x_s = y_s = \pm|g|^{-1/2}$ cancel, whereas for the off-diagonal elements they reinforce each other to give

$$\text{Im } G_{12} = \pi^{3/2}|g|^{-1/2}\,\mathrm{e}^{-2/|g|}, \tag{2.26}$$

in agreement with equation (2.18).

In the next section we will carry out the corresponding calculation for the full problem.

3. Evaluation of the field-theoretic Green function

3.1. The saddle point

We search for the maximum of $\exp(-\int \psi^2)$ under the constraint

$$\det \Gamma(\psi, m) = 0, \tag{3.1}$$

where

$$\Gamma(\psi, m) = (\tfrac{1}{2}m^2 + \mathrm{i}\sqrt{g}\,\psi(x))\delta(x-x') - \tfrac{1}{2}\nabla^2. \tag{3.2}$$

The variational equation reads

$$2\psi(x) + (\rho \times \delta/\delta\psi(x)) \det \Gamma(\psi, m) = 0. \tag{3.3}$$

From the well-known relation

$$\det \Gamma(\psi, m) = \exp(\mathrm{tr} \ln \Gamma(\psi, m)), \tag{3.4}$$

the functional derivative is easily evaluated; equation (3.3) becomes

$$2\psi(x) + \rho\, \mathrm{i}\sqrt{g}\, \langle x|\Gamma^{-1}(\psi, m)|x\rangle \det \Gamma(\psi, m) = 0. \tag{3.5}$$

Now if we write $\det \Gamma(\psi, m)$ as a product of the eigenvalues λ_i^m, $i = 0, 1, 2, \ldots$ of the matrix Γ and evaluate equation (3.5) in the limit $\lambda_0^m \to 0$, we find

$$\psi(x) + \mathrm{i}\sqrt{g}\frac{\rho}{2}\prod_{i=1}^{\infty} \lambda_i^m (\phi_0^m(x))^2 = 0. \tag{3.6}$$

Here $\phi_0^m(x)$ is the normalised eigenstate corresponding to $\lambda_0^m = 0$:

$$\Gamma(\psi, m)\phi_0^m = 0. \tag{3.7}$$

Equations (3.6) and (3.7) give the self-consistency condition

$$\left(-\frac{\nabla^2}{2} + \frac{m^2}{2} + g\frac{\rho}{2}\prod_{i=1}^{\infty} \lambda_i^m (\phi_0^m(x))^2\right)\phi_0^m(x) = 0. \tag{3.8}$$

A non-trivial solution to equation (3.8) is found if

$$\phi_0^m(x) = A^{-1} m^{d/2} \chi(m(x-a)), \tag{3.9}$$

as $\chi(z)$ satisfies the instanton equation

$$(-\nabla^2 + 1)\chi(z) - \chi^3(z) = 0 \tag{3.10}$$

if the Lagrange multiplier ρ is chosen according to

$$1 = -g\rho \prod_{i=1}^{m} \lambda_i^m A^{-2} m^{d-2}. \tag{3.11}$$

The normalisation constant A is given by

$$A = \left(\int \mathrm{d}^d z \chi^2(z)\right)^{1/2}. \tag{3.12}$$

From equations (3.6), (3.9) and (3.12) the field ψ_s at the saddle point is found to be

$$\psi_s(x) = (\mathrm{i}/2\sqrt{g}) m^2 \chi^2(m(x-a)), \tag{3.13}$$

where $\chi(m(x-a))$ describes the instanton centred at a.

3.2. Fluctuations around the saddle point

It is now convenient to exhibit the mass dependence of the problem explicitly. The eigenvalues λ_i^m and eigenfunctions ϕ_i^m of the operator $\Gamma(\psi_s, m)$ are related to the corresponding λ_i and ϕ_i of the theory with mass $m = 1$ by

$$\lambda_i^m = m^2\lambda_i, \qquad \phi_i^m = m^{d/2}\phi_i(mx). \tag{3.14}$$

With this choice the ϕ_i^m form an orthonormal set. We can expand the field ψ in a complete orthonormal set of functions $\bar{\psi}_i^m$:

$$\bar{\psi}_i^m(x) = m^{d/2}\bar{\psi}_i(mx). \tag{3.15}$$

We write

$$\psi = \psi_s + \delta\psi^m, \qquad \delta\psi^m = \sum_{i=0}^{\infty} a_i\bar{\psi}_i^m \tag{3.16}$$

and choose $\bar{\psi}_0$ to be proportional to χ^2,

$$\bar{\psi}_0(x) = I^{-1}\chi^2(x), \tag{3.17}$$

where the normalisation

$$I = \left(\int \mathrm{d}^d x\, \chi^4(x)\right)^{1/2}. \tag{3.18}$$

Following the argument outlined in § 2 we search for the surface in function space determined by the vanishing eigenvalue λ_0. Perturbation theory in $\delta\psi^m$ yields

$$\lambda = \mathrm{i}\sqrt{g}\sum_i a_i\langle\phi_0^m|\bar{\psi}_i^m|\phi_0^m\rangle - (\mathrm{i}\sqrt{g})^2\sum_{i,j} a_i a_j \sum_{k\neq 0}(\lambda_k^m - \lambda)^{-1}\langle\phi_0^m|\bar{\psi}_i^m|\phi_k^m\rangle\langle\phi_k^m|\bar{\psi}_j^m|\phi_0^m\rangle + \cdots. \tag{3.19}$$

Equations (3.9) and (3.17) together with the orthogonality of the set $\bar{\psi}_i^m$ give

$$\langle\phi_0^m|\bar{\psi}_i^m|\phi_0^m\rangle = m^{d/2}A^{-2}I\delta_{i0}, \tag{3.20}$$

and therefore

$$\lambda = -|g|^{1/2}m^{d/2}A^{-2}Ia_0 - |g|\sum_{i,j} a_i a_j \sum_{k\neq 0}(\lambda_k^m - \lambda)^{-1}\langle\phi_0^m|\bar{\psi}_i^m|\phi_k^m\rangle\langle\phi_k^m|\bar{\psi}_j^m|\phi_0^m\rangle + \cdots. \tag{3.21}$$

Eliminating a_0 in favour of λ we find the Jacobian of the transformation given by

$$\left.\frac{\partial a_0}{\partial\lambda}\right|_{a_1,a_2,\ldots\text{fixed}} = -\frac{A^2}{|g|^{1/2}m^{d/2}I}(1 + \mathrm{O}(|g|^{1/2})). \tag{3.22}$$

On the surface $\lambda = 0$ the amplitude a_0 is determined by

$$a_0 = -|g|^{1/2}m^{d/2-2}\frac{A^2}{I}\sum_{i,j\neq 0} a_i a_j \sum_{k\neq 0}\lambda_k^{-1}\langle\phi_0|\bar{\psi}_i|\phi_k\rangle\langle\phi_k|\bar{\psi}_j|\phi_0\rangle + \mathrm{O}(|g|). \tag{3.23}$$

Then expanding the argument of the exponential $\exp(-\int\psi_x^2)$ around the saddle point, allowing only variations in the surface $\lambda = 0$, we find

$$\int \mathrm{d}^d x\,\psi^2(x) = \frac{I^2}{4|g|}m^{4-d} + \langle\delta\psi|\mathcal{M}|\delta\psi\rangle, \tag{3.24}$$

where

$$\langle \delta\psi | \mathcal{M} | \delta\psi \rangle = \int \mathrm{d}^d x \, \mathrm{d}^d x' \, \delta\psi(x)(\delta(x-x') - \chi(x)\Gamma^{-1}(\psi_s, m=1)\chi(x'))\delta\psi(x'). \tag{3.25}$$

The allowed variations $\delta\psi$ are orthogonal to $\bar{\psi}_0$.

Due to translational invariance the matrix $\mathcal{M}$ has d zero eigenvalues with the corresponding normalised eigenfunctions $(\partial/\partial a_\mu)\psi_s(x-a)|_{a=0}$, $\mu = 1, \ldots, d$. Changing variables in the usual way (see e.g. Brézin *et al* 1977 and references therein) we find the Jacobian

$$J = \left((1/d) \int \mathrm{d}^d x (\nabla \psi_s(x))^2 \right)^{d/2} \tag{3.26}$$

$$= [-(1/4g) m^{6-d} I_1^2]^{d/2}, \tag{3.27}$$

where

$$I_1^2 = (1/d) \int \mathrm{d}^d z (\nabla \chi^2(z))^2. \tag{3.28}$$

Parametrising the functional integral (2.2) in terms of λ, a and $\delta\psi$ we may write down an explicit expression for Im $G(x_1, x_2)$,

$$\text{Im } G(x_1, x_2) = \pi \int \mathrm{d}\lambda \, \delta(\lambda) |\partial a_0 / \partial \lambda| \int \mathrm{d}^d a \, J \pi^{-(d+1)/2} \phi_0^m(x_1 + a) \phi_0^m(x_2 + a)$$

$$\times \int \mathrm{D}[\delta\psi] \exp(-\langle \delta\psi | \mathcal{M} | \delta\psi \rangle - \tfrac{1}{4} I^2 |g|^{-1} m^{4-d}), \tag{3.29}$$

where now $\delta\psi$ is orthogonal to $\bar{\psi}_0$ and $(\partial/\partial a_\mu)\psi_s$. Using equations (3.9), (3.22), (3.27) and performing the integral over λ, the m-dependence of Im G is made explicit,

$$\text{Im } G(x_1, x_2) = \pi^{(1-d)/2} (|g| m^{d-4})^{-(d+1)/2} m^{d-2} (\tfrac{1}{2} I_1)^d I^{-1}$$

$$\times C \, \mathrm{e}^{-j^2/4|g|m^{4-d}} \int \mathrm{d}^d a \, \chi(mx_1 - a)\chi(mx_2 - a), \tag{3.30}$$

where

$$C = (\det \mathcal{M})^{-1/2}. \tag{3.31}$$

As can be seen from the derivation, this result is valid when the dimensionless coupling constant $|g| m^{d-4}$ is small; it confirms equation (27) of Cardy (1978). As noted in § 2 for the random electron problem $\frac{1}{2} m^2 = -E$; therefore equation (3.30) implies a density of states in the tail of the band

$$\rho(E) = \rho_0 |g|^{-(d+1)/2} (-E)^{d(5-d)/4} \exp[-I^2(-2E)^{2-d/2}/4|g|]. \tag{3.32}$$

We should point out that the stability matrix $\mathcal{M}$ occurring in our method differs from that of the $n \to 0$ Landau–Ginzburg theory. However, when the determinant of $\mathcal{M}$ is calculated (see Appendix) the constant appearing in equation (3.30) coincides with the usual result.

We note that we have ignored renormalisation problems which are to be handled by standard methods (Zinn-Justin 1978; private communication). As first discussed by Halperin and Lax (1967), the constant C is singular. This divergence is an artifact of the white-noise model and can be avoided by introducing a small but finite range Λ^{-1} of the correlations of the potential fluctuations. In the regularised model, C for $d > 2$ is found

to have a contribution $\exp[B(\Lambda/m)^{d-2}]$, where B is a numerical constant. Evidently this contribution diverges in the white-noise limit $\Lambda \to \infty$. For $2 < d < 4$ this artifact can be cured by a simple mass renormalisation (Zinn-Justin 1978; private communication). We write $m^2 = m_R^2 + |g| I^{-2} B \Lambda^{d-2}[4/(4-d)]$ and we expand the exponent in powers of g. Working consistently to order g^0 we find a counterterm $-B(\Lambda/m_R)^{d-2}$ which cancels the divergency in C to the order g^0 considered. The effect is to shift the energy scale such that $E = m_R^2/2$ is measured from the mobility edge (Halperin and Lax 1967, Edwards 1978), and to substitute C by a finite renormalised constant. With this reinterpretation, our results are finite. For $d = 2$ more complicated behaviour may result (Thouless and Elzain 1978).

4. Conclusions

We show that direct evaluation of the Green function of the random electron or polymer problem by a steepest-descents method yields results identical to those obtained earlier as the $n = 0$ limit of Landau–Ginzburg field theory. The essential feature of our approach lies in the observation that only a restricted set of potential functions contributes to the imaginary part of $G(x_1, x_2)$, and therefore the variational procedure has to be restricted to this subset.

Our calculation verifies that the correct asymptotic information has been used in calculating the critical exponents for the polymer system and supports the calculation of the density of states in the band tail for the random electron problem which has been presented by Cardy. Like the $n \to 0$ Landau–Ginzburg field theory it can be used to evaluate other quantities of interest. For instance consider the expression

$$\overline{|\mathcal{G}(E)|^2} = \lim_{\delta \to 0} \delta \int \mathrm{D}[\psi] \exp\Big(- \int \mathrm{d}^d x \, \psi^2(x) \Big) \mathcal{G}(x_1, x_2, E + \mathrm{i}\delta) \mathcal{G}(x_1, x_2, E - \mathrm{i}\delta), \tag{4.1}$$

which is non-zero if the states are localised (Anderson 1958). Again a non-vanishing contribution is caused only by those potentials ψ which yield a singular operator $\Gamma^{-1}(\psi, (-2E)^{1/2})$; within our method the variational problem is identical to that solved above. We can therefore immediately write down the result

$$\overline{|\mathcal{G}(E)|^2} = \lim_{\eta \to 0} \eta \int \mathrm{d}\lambda \, \frac{1}{(\lambda + \mathrm{i}\eta)(\lambda - \mathrm{i}\eta)} \left| \frac{\partial a_0}{\partial \lambda} \right| J$$

$$\times C \exp(-I^2 m^{4-d}/4|g|) \int \mathrm{d}^d a \, \phi_0^2(x_1 + a) \phi_0^2(x_2 + a), \tag{4.2}$$

which gives

$$\overline{|\mathcal{G}(E)|^2} = \text{constant} \times |g|^{-(d+1)/2} (-E)^{d(7-d)/4} \exp[-I^2(-2E)^{2-d/2}/4|g|]$$

$$\times \int \mathrm{d}^d a \, \chi^2(m x_1 + a) \chi^2(m \chi_2 + a). \tag{4.3}$$

Again this result coincides with that given by Cardy. We note, however, that, unlike Cardy, in this derivation we do not have the problem of solving two coupled instanton equations. Two coupled equations will occur only if we ask for the average of a product of two Green functions evaluated at *two different* energies E and $E + \omega$. This problem, which is related to the AC conductivity of a random electron system, is currently under investigation.

Acknowledgments

One of us (A H) wishes to thank the Alexander von Humboldt Foundation for a Senior US Scientist Award and his colleagues at the Institue for Theoretical Physics, Heidelberg, particularly Professor H Horner, Dr L Schäfer and Professor F J Wegner for their hospitality.

Appendix. Evaluation of the determinant of $\mathcal{M}$

First the operator $\Gamma(\psi_s, m = 1)$ is regularised so that it has no zero eigenvalues:

$$\Gamma(\psi_s, m = 1) = -\tfrac{1}{2}\nabla^2 + \tfrac{1}{2} - \tfrac{1}{2}\chi^2(x), \tag{A1}$$

$$\Gamma_\epsilon = \Gamma(\psi_s, m = 1) - \tfrac{1}{2}\epsilon\chi^2(x). \tag{A2}$$

We know that both χ^2 and $(\partial/\partial x_\mu)\chi^2$ are eigenfunctions of $\mathcal{M}_\epsilon$:

$$\mathcal{M}_\epsilon = I - \chi\Gamma_\epsilon^{-1}\chi. \tag{A3}$$

Let

$$\mathcal{M}_\epsilon\chi^2 = E_0\chi^2; \tag{A4}$$

then

$$(1 - E_0)\Gamma_\epsilon\chi = \chi^3. \tag{A5}$$

But from equation (A2)

$$\Gamma_\epsilon\chi = -\tfrac{1}{2}\epsilon\chi^3; \tag{A6}$$

therefore:

$$E_0 = 1 + 2/\epsilon. \tag{A7}$$

Similarly, if we write

$$\mathcal{M}_\epsilon(\partial/\partial x_\mu)\chi^2 = E_1(\partial/\partial x_\mu)\chi^2, \tag{A8}$$

we find

$$E_1 = -\epsilon/2. \tag{A9}$$

Now χ is approximately an eigenfunction of Γ_ϵ with energy

$$\hat{E}_0 = \langle\chi| -\tfrac{1}{2}\epsilon\chi^2|\chi\rangle/\langle\chi|\chi\rangle = -\tfrac{1}{2}\epsilon A^{-2}I^2. \tag{A10}$$

Similarly $\partial\chi/\partial x_\mu$ is approximately an eigenfunction of $\Gamma_\epsilon - \chi^2$, with energy

$$\hat{E}_1 = -\tfrac{1}{8}\epsilon I_1^2\langle\partial\chi/\partial\chi_\mu|\partial\chi/\partial x_\mu\rangle^{-1}. \tag{A11}$$

We now have

$$\begin{aligned}\det\mathcal{M} &= \lim_{\epsilon\to 0} E_0^{-1}E_1^{-d}\det(I - \chi\Gamma_\epsilon^{-1}\chi)\\ &= \lim_{\epsilon\to 0} E_0^{-1}E_1^{-d}\det(\Gamma_\epsilon - \chi^2)(\det\Gamma_\epsilon)^{-1}\\ &= \lim_{\epsilon\to 0} (E_0\hat{E}_0)^{-1}(\hat{E}_1/E_1)^d\det'(\Gamma - \chi^2)/\det'\Gamma,\end{aligned} \tag{A12}$$

where the determinants $\mathrm{Det}'(\Gamma-\chi^2)$ and $\mathrm{Det}'\,\Gamma$ have to be taken in the space orthogonal to $\partial\chi/\partial x_\mu$ and χ respectively. Using equations A7, A9, A10 and A11 we find

$$\det \mathcal{M} = -\frac{A^2}{I^2}\left(\frac{I_1^2}{4}\left\langle\frac{\partial\chi}{\partial x_\mu}\middle|\frac{\partial\chi}{\partial x_\mu}\right\rangle^{-1}\right)^d \frac{\mathrm{det}'(\Gamma-\chi^2)}{\mathrm{det}'\,\Gamma}. \tag{A13}$$

Using this result in equation (3.30) we find the same constant as in the $n\to 0$ Landau–Ginzburg theory.

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